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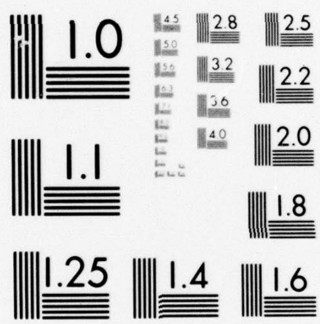

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In this paper, we prove the truth of the following decomposition of the solutions to the lossless wave equation in layered media; the complete output from a K-layer media system, which is comprised of the superposition of primaries, secondaries, tertiaries, etc., can be obtained from a single state space model of order 2K - the complete model - or from an infinite number of models, each of order 2K, the output of the first of which is just the primaries, the output of the second of which is just the secondaries, etc. This decomposition of the solution to the lossless wave equation into physically meaningful constituents

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(i. e., primaries, secondaries, etc.) is called a canonical Bremmer Series decomposition, after Bremmer, who in 1951 established a similar decomposition.

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A CANONICAL BREMMER SERIES DECOMPOSITION OF SOLUTIONS TO THE LOSSLESS WAVE EQUATION IN LAYERED MEDIA

Jerry M. Mendel
University of Southern California, California

Abstract

In this paper, we prove the truth of the following decomposition of the solutions to the lossless wave equation in layered media: the complete output from a K-layer media system, which is comprised of the superposition of primaries, secondaries, tertiaries, etc., can be obtained from a single state space model of order 2K - the complete model - or from an infinite number of models, each of order 2K, the output of the first of which is just the primaries, the output of the second of which is just the secondaries, etc. This decomposition of the solution to the lossless wave equation into physically meaningful constituents (i.e., primaries, secondaries, etc.) is called a canonical Bremmer Series decomposition, after Bremmer, who in 1951 established a similar decomposition.

In many geophysical situations, where reflection coefficients are quite small, the decomposition can be truncated after secondaries or tertiaries; hence, it also represents a way to approximate the solution to the wave equation.

I. Introduction

Nahi and Mendel (Ref. 1) have developed state space equations for a model of a horizontally stratified nonabsorptive earth with vertically travelling plane compressional waves. Their model is based on a ray theory solution to the lossless wave equation in layered media. As such, it is not limited just to layered earth models; however, since our interest is in the seismic area, our discussions will be in the context of such models. Mendel (Ref. 1) conjectured, and demonstrated by examples, that "the complete output from a K-layer media system, which is comprised of the superposition of primaries, secondaries, tertiaries, etc., can be obtained from a single model of order 2K - the complete model - or from an infinite number of models, each of order 2K ...," the output of the first of which is just the primaries, the output of the second of which is just the secondaries, etc. We refer to this decomposition of the complete solution to the lossless wave equation into physically meaningful components (i.e., primaries, secondaries, etc.) as a canonical decomposition.

In this paper, after we review the state space model of a K-layer media, we prove the truth of the canonical decomposition and illustrate its use by means of a simulation example. We show that, whereas the complete model (i.e., the model that generates primaries and all multiples for the K-layer media) has both poles and zeros, similar to an ARMA model, the partial residual models (such as the primaries model, which generates only primary reflections, and the secondaries models, which generates only the secondary reflections) only have zeros, similar to MA models. As such, the partial residual models are much easier to compute with than the complete model, since they are open-loop models, whereas the complete model is a feedback model. In many geophysical situations, where reflection coefficients are quite small, the canonical decomposition can be truncated after secondaries or tertiaries; hence, it also represents a way to approximate the complete solution to the wave equation.

After the author had developed and proved the canonical decomposition, he discovered that the idea of such a decomposition, as well as a collection of integral equations for accomplishing it, were presented by Bremmer (Ref. 2) in 1951. In fact, such a decomposition is referred to as a Bremmer Series (Refs. 3 and 4); hence, we shall refer to our canonical decomposition as a Bremmer Series decomposition. Our approach to the development of this decomposition is quite different from Bremmer's. We shall have more to say about this in Section VII.

A system of K-layered media is depicted in Figure 1. We adopt the convention of calling the layer below layer K the basement. Each layer is characterized by its one way travel time τ_i, velocity, V_i, and normal incidence reflection coefficients r_i ($i = 1, 2, \ldots, K$). Additionally, interface-0 denotes the surface and is characterized by reflection coefficient r_0. In Figure 1, m(t) and y(t) denote the input (e.g., seismic source signature from

dynamite, airgun, etc.) to the layered earth system* which is applied at interface-0, and the output (i.e., ideal seismogram) of the system which is observed at the surface, respectively.

As in Ref. 5, we shall find it convenient to draw ray diagrams with time displacement along the horizontal axis, so that the rays appear to be at non-normal incidence and so do not overlap one another.

Figure 2 depicts primary and multiple reflections for a 2 layer media, and illustrate the very complicated internal behavior of the two-layer system, and, that y(t) has components which occur at specific values of t. The primary reflection components of y(t), denoted $y_1(t)$, are depicted in Figure 3a, whereas the secondary reflection components of y(t), denoted $y_2(t)$, are depicted in Figure 3b.

Observe, from this simple example, that primary reflections result from exactly one reflection off an interface in a layered media system; hence, primary reflection components, $y_1(t)$ include only those terms which, neglecting the effects of two-way transmission factors, are proportional to linear functions of the reflection coefficients [e.g., $r_0(1-r_1^2)\,m(t-2\tau_1) \propto r_0 m(t-2\tau_1)$, where $\propto$ denotes proportionality]. Observe, also, that secondary reflections result from exactly three reflections off of interfaces within a layered media system; hence, secondary reflection components, $y_2(t)$, of y(t) include only those terms which, neglecting the effects of two-way transmission factors, are proportional to cubic functions of the reflection coefficients. In general, n-ary reflections result from exactly 2n - 1 reflections off of interfaces within a layered media system; hence, n-ary reflection components, $y_n(t)$, of y(t) include only those terms which, neglecting the effects of two-way transmission factors, are proportional to products of 2n - 1 reflection coefficients.

By thinking of y(t) as the superposition of its constituents - primaries, secondaries, etc., - we can write it as

$$y(t) = \sum_{j=1}^{\infty} y_j(t) \quad . \qquad (1)$$

Equation (1) is the Bremmer Series decomposition of y(t). We shall demonstrate that such a decomposition is possible, and, shall provide state space models for obtaining the constituents, $y_1(t), y_2(t), \ldots$. Our state space models can be thought of as numerical procedures (algorithms) for obtaining the constituents of y(t).

Let us express each reflection coefficient in terms of a small number, say ε; that is to say, given $r_0, r_1, \ldots, r_k$ and ε, numbers r_0', $r_1', \ldots, r_k'$ can be determined such that

$$r_j = \varepsilon r_j' \quad , \quad j = 0, 1, \ldots, k \qquad (2)$$

From our preceding discussion on the proportionality of $y_i(t)$ to products of 2i - 1 reflection coefficients, we see that

$$y_i(t) = \varepsilon^{2i-1} y_i'(t) \quad , \quad i = 1, 2, \ldots \qquad (3)$$

which means that y(t) in Eq. (1) may be expressed as the following power series in ε:

$$y(t) = \varepsilon y_1'(t) + \varepsilon^3 y_2'(t) + \varepsilon^5 y_3'(t) + \ldots + \varepsilon^{2i-1} y_i'(t) + \ldots \qquad (4)$$

This equation suggests the possibility of truncating the Bremmer Series decomposition after a small number of terms (unless the amplitude of $y_i'(t)$ is comparable to ε^{2i-1}, which is possible in certain applications). It also provides us with a method for proving that our partial residual state equation models generate just the primaries, or just the secondaries (these models are described in Sections III and IV), etc. We shall prove that the output of our primaries model, $y_1(t)$, is proportional to ε, that the output of our secondaries model, $y_2(t)$, is proportional to ε^3, etc.

In Section II we review the state equation model described by Nahi and Mendel (Ref. 1) for generating the complete response. In Section III we develop and prove the validity of a state equation primaries model. In Section IV we develop and indicate the proof of validity of a state equation secondaries model. A complete statement of the canonical Bremmer Series decomposition is given in Section V. An example which demonstrates the decomposition for a Bright Spot application is given in Section VI. Relationships between our results and Bremmer's, as well as conclusions, are given in Section VII.

II. A State Equation Model for Complete Response

There are (at least) two different ways for obtaining a state equation model for the complete response, y(t), of a K-layer media system (Ref. 1). The brief derivation of this model, given below, is based on a starting point which should be quite familar to all exploration geophysicists. It is the ray diagram, depicted in Figure 4, which is taken from Ref. 5. Symbols u_k and d_k' denote the upgoing and downgoing waves in the k-th layers, respectively; and, we adopt the convention that waves at the top of a layer occur at present time, t. To develop the

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* In a marine environment, layer 1 can be taken as water. For the purposes of the present paper, we assume that both the source and the sensor are located at the surface. The generalization of our results to subsurface source and sensor is straightforward.

state equation model we direct our attention at the intersection point of the ray diagram and apply superposition to obtain the following equations for signals $u_k(t-\tau_k)$ and $d'_{k+1}(t)$, which leave that point:

$$u_k(t+\tau_k) = r_k d'_k(t-\tau_k) + (1-r_k)u_{k+1}(t) \quad (5a)$$

$$d'_{k+1}(t) = (1+r_k)d'_k(t-\tau_k) - r_k u_{k+1}(t) \quad (5b)$$

These equations are applicable for $k = 1, 2, \ldots, K-1$. At the surface (Figure 5a), we obtain

$$y(t) = r_0 m(t) + (1-r_0)u_1(t) \quad (6)$$

$$d'_1(t) = (1+r_0)m(t) - r_0 u_1(t) \quad (7)$$

and, at the K^{th} interface, we assume that $u_{K+1}(t) = 0$, to obtain (Figure 5b)

$$u_K(t+\tau_K) = r_K d'_K(t-\tau_K) \quad (8)$$

$$d'_{K+1}(t) = (1+r_K)d'_K(t-\tau_K) \quad (9)$$

Signal $y(t)$ in Eq. (6) is the measurable system output. Signal $d'_{K+1}(t)$ is also a system output; but, since it cannot be measured, we shall ignore it in following analyses.

It is convenient to group Eqs. (5a), (5b), (7) and (8) in a layer ordering, as follows:

$$\begin{aligned}
d'_1(t) &= -r_0 u_1(t) + (1+r_0)m(t) \\
u_1(t+\tau_1) &= r_1 d'_1(t-\tau_1) + (1-r_1)u_2(t) \\
&\left.\begin{aligned}
d'_j(t) &= (1+r_{j-1})d'_{j-1}(t-\tau_{j-1}) - r_{j-1}u_j(t) \\
u_j(t+\tau_j) &= r_j d'_j(t-\tau_j) + (1-r_j)u_{j+1}(t)
\end{aligned}\right\} j=2,3,\ldots,K-1 \\
d'_K(t) &= (1+r_{K-1})d'_{K-1}(t-\tau_{K-1}) - r_{K-1}u_K(t) \\
u_K(t+\tau_K) &= r_K d'_K(t-\tau_K)
\end{aligned} \quad (10)$$

This system of 2K equations is not in useful state equation format, yet, since signals in its left-hand side occur at t and delayed times, and signals on the right-hand side occur at t, $t-\tau_{j-1}$ and $t-\tau_j$. In order to put Eq. (10) into useful state equation format, let

$$d_j(t) \triangleq d'_j(t-\tau_j) \quad (11)$$

for all $j = 1, 2, \ldots, K$. Observe, from Figure 4, that the downgoing states $d_j(t)$ occur at the bottom of a layer. Equation (10) becomes

$$\begin{aligned}
d_1(t+\tau_1) &= -r_0 u_1(t) + (1+r_0)m(t) \\
u_1(t+\tau_1) &= r_1 d_1(t) + (1-r_1)u_2(t) \\
&\left.\begin{aligned}
d_j(t+\tau_j) &= (1+r_{j-1})d_{j-1}(t) - r_{j-1}u_j(t) \\
u_j(t+\tau_j) &= r_j d_j(t) + (1-r_j)u_{j+1}(t)
\end{aligned}\right\} j=2,3,\ldots,K-1 \\
d_K(t+\tau_K) &= (1+r_{K-1})d_{K-1}(t) - r_{K-1}u_K(t) \\
u_K(t+\tau_K) &= r_K d_K(t)
\end{aligned} \quad (12)$$

By means of transformation (11) each pair of equations in (10) now only involves two time points, $t+\tau_j$ and t. Equations (12) and (6) together represent the state equation model for the complete output $y(t)$; hence, they are referred to in the sequel as the complete model.

Comment. Equations (5a), (5b), (7) and (8), for uniform travel times (i.e., $\tau_1 = \tau_2 = \ldots = \tau_K \triangleq \tau$), represent the starting point for the well-known transfer function models of a K-layer media (Refs. 5, 6 and 7, for example). Starting with those same equations, we have taken a path different from the one which leads to transfer function models; our path leads to time-domain, state space equations. In the special case when $\tau_1 = \tau_2 = \ldots = \tau_K \triangleq \tau$, Eq. (12) can be reduced to a collection of finite-difference equations, by choosing $t = k\tau$. We shall explore the implications of taking this other path on such problems as computation and deconvolution in a forthcoming paper. ■

In order to develop additional insight into the structure of our state space model, it is instructive to portray that model in signal-flow graph (SFG) form. Because all important features of our model can be observed from the nature of a two-layer example, we shall present our SFG for that case, leaving generalizations to the reader.

For purposes of constructing a SFG it is useful to rewrite Eq. (12), for $K = 2$, as follows:

$$\begin{aligned}
d_1(t) &= -r_0 u_1(t-\tau_1) + (1+r_0)m(t-\tau_1) \\
d_2(t) &= (1+r_1)d_1(t-\tau_2) - r_1 u_2(t-\tau_2) \\
u_2(t) &= r_2 d_2(t-\tau_2) \\
u_1(t) &= r_1 d_1(t-\tau_1) + (1-r_1)u_2(t-\tau_1)
\end{aligned}$$

The ordering of the state - d_1, d_2, u_2, u_1 - is a computational ordering, because that is precisely the order in which we would solve these four equations. A SFG based on these equations and Eq. (6) is depicted in Figure 6. For simplicity in drawing the SFG, we do not show

the explicit dependence of node signals on time; and, we use operators z_1 and z_2 to denote τ_1 and τ_2 sec. time delays, respectively [i.e., $z_1 m = m(t-\tau_1)$]. The feedback nature of the layered media system is clearly evident from the SFG. Three feedback loops are present; the inner loop being associated with layer 2, the other loops being associated with layers 1 and 2.

The feedback nature of the complete model indicates that it has zeros as well as poles; hence, it is similar to an autoregressive moving average (ARMA) model. It is the feedback nature of the complete model which produces the multitude of multiple reflections (see Figure 2, for example).

III. Primaries Model

Conceptually, the primary reflections (i.e., primatires) are easily obtained directly from ray diagrams (Figure 3a, for example); however, that approach does not lead to a dynamical model which can be associated with the primaries. In this section we show how to modify the complete model in a very simple manner so as to obtain a state space model which generates only the primaries. We refer to this model either as a primary reflection model or a primaries model.

A. Conjectured Primaries Model

We conjecture that state equations for a primaries model are obtained from Eq. (12) by deleting the term $-r_{j-1}\, u_j(t)$ in the equations for $d_j(t+\tau_j)$, $j=1,2,\ldots,K$. Our reason for deleting these terms is based on the following observation: a multiple reflection can occur only when an upgoing wave reflects off of the top of a layer. Observe, in Figure 4, that the contribution due to $u_{k+1}(t)$ in $d'_{k+1}(t)$ represents just such a reflection; hence, we delete these components from the complete model in our primaries model, whose equations are given below.

$$d_{1,1}(t+\tau_1) = (1+r_0)\, m(t)$$

$$u_{1,1}(t+\tau_1) = r_1\, d_{1,1}(t) + (1-r_1)\, u_{1,2}(t)$$

$$\left.\begin{aligned} d_{1,j}(t+\tau_j) &= (1+r_{j-1})\, d_{1,j-1}(t) \\ u_{1,j}(t+\tau_j) &= r_j\, d_{1,j}(t) \\ &\quad + (1-r_j)\, u_{1,j+1}(t) \end{aligned}\right\} j=2,3,\ldots,K-1$$

$$d_{1,K}(t+\tau_K) = (1+r_{K-1})\, d_{1,K-1}(t)$$

$$u_{1,K}(t+\tau_K) = r_K\, d_{1,K}(t) \qquad (13)$$

and

$$y_1(t) = r_0\, m(t) + (1-r_0)\, u_{1,1}(t) \qquad (14)$$

In these equations, $d_{1,j}$, $u_{1,j}$, and y_1 denote primary downgoing and upgoing state and the primary reflection portion of the complete output, respectively.

At this point Eqs. (13) and (14) are conjectural in nature; i.e., that they do give just the primaries remains to be proven. We do this in Paragraph B below. They have been validated by ray tracing arguments for a three-layer example in Ref. 1.

Before validating our model, we wish to point out an important property of the primaries model: its SFG has no feedback paths. We illustrate this for a two-layer media in Figure 7. This property causes $y_1(t)$ to be comprised of a finite number of terms - $K+1$, for a K-layered media - and enables $y_1(t)$ to be computed in a very straightforward manner. The open-loop nature of the primaries model indicates it only has zeros; hence, it is similar to a moving average (MA) model.

B. Validation of Primaries Model

We shall show that $y_1(t)$, as given by Eq. (14), is proportional to $\varepsilon\, g_1(t)$. Function $g_1(t)$ is not equal to $y_1(t)$ in Eq. (3), because we use proportionality, rather than equality, arguments in our proof. Now, it is possible that we may not have accounted for all terms which are proportional to ε in our conjectured primaries model; hence, we must examine the residual between $y(t)$ and $y_1(t)$. We shall show that that residual contains no terms which are proportional to ε, hence, $y_1(t)$ contains all the terms which are proportional to ε, and, therefore, our conjectured primaries model is a valid primaries model.

1. $y_1(t)$ Proportional to ε

We wish to show that $y_1(t)$, as given by Eq. (14), is proportional to $\varepsilon\, g_1(t)$. To begin, we rewrite Eqs. (13) and (14) in proportionality format, using a computational ordering of the states, where we assume $(1 \pm \varepsilon r')\, f(t) \propto f(t)$. By this technique, we are absorbing all the two-way transmission factors, like $(1-r_1^2) = (1-\varepsilon^2 r_1'^2)$, which appear in $y_1(t)$, in the proportionality arguments. Equality arguments can also be used; but, the algebra associated with them is more complicated.

$$d_{1,1}(t+\tau_1) \propto m(t)$$

$$d_{1,j}(t+\tau_j) \propto d_{1,j-1}(t)\ , \quad j=2,3,\ldots,K-1$$

$$u_{1,K}(t+\tau_K) \propto \varepsilon\, d_{1,K}(t)$$

$$u_{1,j}(t+\tau_j) \propto \varepsilon\, d_{1,j}(t) + u_{1,j+1}(t)\ ,$$
$$j=1,2,\ldots,K-1 \qquad (15)$$

and

$$y_1(t) \propto \varepsilon\, m(t) + u_{1,1}(t) \; . \tag{16}$$

Obviously, we must show that $u_{1,1}(t)$ is linear in ε. From Eq. (15), it follows that

$$d_{1,2}(t+\tau_2) \propto d_{1,1}(t) \propto m(t-\tau_1) \triangleq a_2(t)$$

$$d_{1,3}(t+\tau_3) \propto d_{1,2}(t) \propto a_2(t-\tau_2) \triangleq a_3(t)$$

$$\cdots$$

$$d_{1,K}(t+\tau_K) \propto d_{1,K-1}(t) \propto a_{K-1}(t-\tau_{K-1}) \triangleq a_K(t) \tag{17}$$

which demonstrates that all primary downgoing states are not proportional to ε, at all. From Eqs. (15) and (17), it also follows, that

$$u_{1,K}(t+\tau_K) \propto \varepsilon\, d_{1,K}(t) \propto \varepsilon\, a_K(t-\tau_K) \triangleq \varepsilon\, b_K(t)$$

$$\begin{aligned} u_{1,K-1}(t+\tau_{K-1}) &\propto \varepsilon\, d_{1,K-1}(t) + u_{1,K}(t) \\ &\propto \varepsilon\, a_{K-1}(t-\tau_{K-1}) + \varepsilon\, b_K(t-\tau_K) \\ &\triangleq \varepsilon\, b_{K-1}(t) \end{aligned}$$

$$\cdots$$

$$\begin{aligned} u_{1,1}(t+\tau_1) &\propto \varepsilon\, d_{1,1}(t) + u_{1,2}(t) \\ &\propto \varepsilon\, m(t-\tau_1) + \varepsilon\, b_2(t-\tau_2) \\ &\triangleq \varepsilon\, b_1(t) \end{aligned} \tag{18}$$

which demonstrates that all primary upgoing states are proportional to ε. Finally, from Eq. (16) and the last equation in (18), we see that

$$y_1(t) \propto \varepsilon\, m(t) + \varepsilon\, b_1(t) \triangleq \varepsilon\, g_1(t) \; , \tag{19}$$

which was to be shown.

2. First Residual Model

We form a residual model between our complete model and primaries model. Let $\mu_{1,j}(t)$, $\delta_{1,j}(t)$, and $s_1(t)$ denote the first residual upgoing and downgoing states, and first residual output, respectively, where $(j = 1, 2, \ldots, K)$

$$\mu_{1,j}(t) = u_j(t) - u_{1,j}(t) \quad , \tag{20a}$$

$$\delta_{1,j}(t) = d_j(t) - d_{1,j}(t) \quad , \tag{20b}$$

and

$$s_1(t) = y(t) - y_1(t) \quad . \tag{21}$$

Subtract the respective equations in Eqs. (12) and (13), and (6) and (14) to obtain the following <u>first residual state equations</u>:

$$\delta_{1,1}(t+\tau_1) = -r_0\, \mu_{1,1}(t) - r_0\, u_{1,1}(t)$$

$$\mu_{1,1}(t+\tau_1) = r_1\, \delta_{1,1}(t) + (1-r_1)\, \mu_{1,2}(t)$$

$$\left.\begin{aligned} \delta_{1,j}(t+\tau_j) &= (1+r_{j-1})\, \delta_{1,j-1}(t) \\ &\quad - r_{j-1}\mu_{1,j} - r_{j-1} u_{1,j}(t) \\ \mu_{1,j}(t+\tau_j) &= r_j\, \delta_{1,j}(t) \\ &\quad + (1-r_j)\, \mu_{1,j+1}(t) \end{aligned}\right\} j = 2, 3, \ldots, K-1$$

$$\delta_{1,K}(t+\tau_K) = (1+r_{K+1})\, \delta_{1,K-1}(t) - r_{K-1}\mu_{1,K}(t) - r_{K-1}\, u_{1,K}(t)$$

$$\mu_{1,K}(t+\tau_K) = r_K\, \delta_{1,K}(t) \tag{22}$$

and

$$s_1(t) = (1-r_0)\, \mu_{1,1}(t) \tag{23}$$

By its very definition, $s_1(t)$ comprises all the multiple reflections for a K-layer media system. Observe that the first residual state equations are driven by the upgoing states from the primaries model.

3. $s_1(t)$ Contains No Terms Proportional to ε

In terms of proportionality, Eqs. (22) and (23) become:

$$\delta_{1,1}(t+\tau_1) \propto -\varepsilon\, \mu_{1,1}(t) - \varepsilon\, u_{1,1}(t)$$

$$\mu_{1,1}(t+\tau_1) \propto \varepsilon\, \delta_{1,1}(t) + \mu_{1,2}(t)$$

$$\left.\begin{aligned} \delta_{1,j}(t+\tau_j) &\propto \delta_{1,j-1}(t) - \varepsilon\, \mu_{1,j}(t) \\ &\quad - \varepsilon\, u_{1,j}(t) \\ \mu_{1,j}(t+\tau_j) &\propto \varepsilon\, \delta_{1,j}(t) + \mu_{1,j+1}(t) \end{aligned}\right\} j = 2, 3, \ldots, K-1$$

$$\delta_{1,K}(t+\tau_K) \propto \delta_{1,K-1}(t) - \varepsilon\, \mu_{1,K}(t) - \varepsilon\, u_{1,K}(t)$$

$$\mu_{1,K}(t+\tau_K) \propto \varepsilon\, \delta_{1,K}(t) \tag{24}$$

and

$$s_1(t) \propto \mu_{1,1}(t) \tag{25}$$

Obviously, we must show that $\mu_{1,1}(t)$ contains no terms which are proportional to ε. For illustrative purposes, we do this for $K = 2$; the generalization for $K > 2$ is straightforward and can be deduced from the topology of the SFG for $K > 2$. A proportionality SFG, in which nodal signals are proportional to incoming signals is depicted in Figure 8.

By SFG algebra or Mason's Reduction Theorem, we obtain the following relationship between $s_1(t)$ and $u_{1,1}(t)$ and $u_{1,2}(t)$:

$$s_1(t) \propto \left\{\frac{-\varepsilon^2 z_1^2 z_2^2 - \varepsilon^2 z_1^2(1+\varepsilon^2 z_2^2)}{\Delta}\right\} u_{1,1}(t) - \left\{\frac{\varepsilon^2 z_1 z_2^2}{\Delta}\right\} u_{1,2}(t) \quad (26)$$

where

$$\Delta = 1 + \varepsilon^2 z_2^2 + \varepsilon^2 z_1^2 z_2^2 + \varepsilon^2 z_1^2 + \varepsilon^4 z_1^2 z_2^2 \quad (27)$$

Observe that ε^2 is common to each term in Eq. (26). Additionally, we showed in Eq. (18) that $u_{1,1}$ and $u_{1,2}$ are proportional to ε; hence, dividing Δ into the numerator operators in Eq. (26), we see that

$$s_1(t) \propto \varepsilon^3 s_1'(t) + \text{other terms} \quad (28)$$

By means of this 2-layer example, we have demonstrated that $s_1(t)$ contains no terms proportional to ε; hence, all terms proportional to ε have been accounted for in $y_1(t)$. This completes the validation of our primaries model.

IV. Secondaries Model

Secondary reflection components of the complete response are much more difficult to obtain from ray diagrams (Figure 3b, for example) than are primary reflections. The main reason for this is that there are many more *secondaries than* primaries. In this section we show how to modify the first residual model in a very simple manner so as to obtain a state space model which generates only the secondaries. We refer to this model either as a secondary reflection model, secondaries model, or first partial residual model.

A. Conjectured Secondaries Model

Compare the first residual state equations (22) with the complete model state equations (12) to observe that they are structurally quite similar. Recall that in the development of the primaries model we deleted the term $-r_{j-1} u_j(t)$ in the equations for $d_j(t+\tau_j)$, because that term represented the signal which reflects off of an interface and returns down into the layered media. We conjecture, therefore, that the term $-r_{j-1}\mu_{1,j}(t)$ in the equations for $\delta_{1,j}(t+\tau_j)$ plays an analogous role to the $-r_{j-1} u_j(t)$ term; hence, we delete the former terms from the first residual model to obtain the following secondaries model (or, first partial residual model):

$$d_{2,1}(t+\tau_1) = -r_0\, u_{1,1}(t)$$

$$u_{2,1}(t+\tau_1) = r_1\, d_{2,1}(t) + (1-r_1)\, u_{2,2}(t)$$

$$\left.\begin{aligned} d_{2,j}(t+\tau_j) &= (1+r_{j-1})\, d_{2,j-1}(t) - r_{j-1}\, u_{1,j}(t) \\ u_{2,j}(t+\tau_j) &= r_j\, d_{2,j}(t) + (1-r_j)\, u_{2,j+1}(t) \end{aligned}\right\} j = 2, 3, \ldots, K-1$$

$$d_{2,K}(t+\tau_K) = (1+r_{K-1})\, d_{2,K-1}(t) - r_{K-1}\, u_{1,K}(t)$$

$$u_{2,K}(t+\tau_K) = r_K\, d_{2,K}(t) \quad (29)$$

and

$$y_2(t) = (1-r_0)\, u_{2,1}(t) \quad (30)$$

In these equations, $d_{2,j}$, $u_{2,j}$, and y_2 denote secondary downgoing and upgoing states and the secondary reflection portion of the complete output, respectively.

Before discussing the validation of this model, we wish to point out that the SFG of the secondaries model has no feedback paths. We illustrate this for a two-layer media in Figure 9. This property causes $y_2(t)$ to be comprised of the requisite number of terms, and enables $y_2(t)$ to be computed in a very straightforward manner. Because the secondaries model only has zeros, it also is similar to a MA model.

B. Validation of Secondaries Model

It can be shown that $y_2(t)$, as given by Eq. (30), is proportional to $\varepsilon^3 g_2(t)$. The proof of this follows the proof in Section III.B.1 that $y_1(t)$ is proportional to ε. First, show that all secondary downgoing states are proportional to ε^2. Then show that all secondary upgoing states are proportional to ε^3. Finally, from Eq. (30), show that $y_2(t) \propto \varepsilon^3 g_2(t)$.

Next, form a residual model between our first residual model and secondaries model. Let $\mu_{2,j}(t)$, $\delta_{2,j}(t)$, and $s_2(t)$ denote the second residual upgoing and downgoing states, and second residual output, respectively, where $(j = 1, 2, \ldots, K)$

$$\mu_{2,j}(t) = \mu_{1,j}(t) - u_{2,j}(t) \quad (31a)$$

$$\delta_{2,j}(t) = \delta_{1,j}(t) - d_{2,j}(t) \quad (31b)$$

and

$$s_2(t) = s_1(t) - y_2(t) \quad (32)$$

The second residual state and output equations can be obtained by subtracting the respective equations in Eqs. (22) and (29), and (23) and (30). These equations will look exactly like the first residual state and output equations in Eqs. (22) and (23), with $\delta_{1,j}, \mu_{1,j}$, and $s_1(t)$ replaced by $\delta_{2,j}, \mu_{2,j}$, and s_2, respectively, and, $u_{1,j}$ (which forces the first residual state equations) replaced by $u_{2,j}$. The second residual state equations are driven by the upgoing states from the secondaries model.

Final validation of the secondaries model is to show that $s_2(t)$ contains no terms which are proportional to ε^3. The proof of this follows the proof in Section III.B.3 that $s_1(t)$ contains no terms proportional to ε, and is left to the reader.

V. Canonical Bremmer Series Decomposition

Our results in the two preceding sections suggest that a complete decomposition of a seismogram [i.e., $y(t)$] into a superposition of outputs from primaries, secondaries, tertiaries, etc., models is possible. We summarize this decomposition in the following:

Theorem. The complete output, $y(t)$, from a K-layer media system can be obtained from a single model of order 2K - the complete model, given by Eqs. (12) and (6) - or from an infinite number of models, each of order 2K, interconnected as shown in Figure 10. The primaries model is defined by Eqs. (13) and (14). The n-aries (or n-1st partial residual) models are defined by the following:

$$d_{n,1}(t+\tau_1) = -r_0\, u_{n-1,1}(t)$$

$$u_{n,1}(t+\tau_1) = r_1\, d_{n,1}(t) + (1-r_1)\, u_{n,2}(t)$$

$$\left.\begin{aligned} d_{n,j}(t+\tau_j) &= (1+r_{j-1})\, d_{n,j-1}(t) \\ &\quad - r_{j-1}\, u_{n-1,j}(t) \\ u_{n,j}(t+\tau_j) &= r_j\, d_{n,j}(t) \\ &\quad + (1-r_j)\, u_{n,j+1}(t) \end{aligned}\right\} j = 2, 3, \dots, K-1$$

$$d_{n,K}(t+\tau_K) = (1+r_{K-1})\, d_{n,K-1}(t) - r_{K-1}\, u_{n-1,K}(t)$$

$$u_{n,K}(t+\tau_K) = r_K\, d_{n,K}(t) \qquad (33)$$

and

$$y_n(t) = (1-r_0)\, u_{n,1}(t) \qquad (34)$$

for $n = 2, 3, \dots$. Additionally, $y(t)$ is given by Eq. (1).

Proof: An inductive proof of this theorem is straightforward, although details are tedious in nature. We have validated the primaries model in Section III, and have validated Eqs. (33) and (34), for $n = 1$, in Section IV. Assume the truth of Eqs. (33) and (34) for $n = n^*$ and show that they are valid for $n = n^*+1$ via the inductive step. We do not give all the details, because they are so similar to the ones in our validation of the secondaries model (and primaries model). To begin, we can show that $y_{n^*+1}(t)$ is proportional to $\varepsilon^{2n^*+1} g_{n^*+1}$. In order to do this, we use the the (assumed) fact that all n^*-ary upgoing states [i.e., $u_{n^*,j}(t)$] are proportional to ε^{2n^*-1}, to show that all (n^*+1)-ary downgoing states are proportional to ε^{2n^*}. We then show that all (n^*+1)-ary upgoing states are proportional to ε^{2n^*+1}; hence, $y_{n^*+1}(t) \propto \varepsilon^{2n^*+1} g_{n^*+1}(t)$. Next, form a residual model between the n^* residual model and the (n^*+1)-ary model. Let $s_{n^*+1}(t)$ denote the (n^*+1) residual output; i.e.,

$$s_{n^*+1}(t) = s_{n^*}(t) - y_{n^*+1}(t) \qquad (35)$$

Finally, show that $s_{n^*+1}(t)$ contains no terms which are proportional to ε^{2n^*+1}. ∎

VI. Example

To illustrate the canonical Bremmer Series decomposition, we have chosen a 3-layer media (Figure 11) which can be associated with a bright spot phenomenon, because of the thin low velocity layer which is sandwiched in between the two thick high velocity layers. We attribute no other geological plausibility to this example. For layer 1, V_1=7,500 ft/sec and ρ_1=2.2 gm/cm^3; for layer 2, V_2=5,500 ft/sec and ρ_2=1.6 gm/cm^3; for layer 3, $V_3 = V_1$ and $\rho_3 = \rho_1$; for the basement, V_4=12,000 ft/sec and ρ_4 was approximated by the $\frac{1}{4}$-power law, $0.23\, V_4^{\frac{1}{4}}$ (Ref. 8).

Figure 12 depicts the complete response as well as the primaries, secondaries and some of the tertiaries (those through 2 seconds). Occasionally, a tertiary and secondary will be coincident so that their sum, which appears in the lower plot, looks different from either constituent (see the last spike, for example). Even more interesting are the large secondaries and tertiaries which occur well into the seismogram. These large components always involve (at least one) reflection off of the surface interface, where r_0 is close to unity; however, that in itself does not explain such large components.

There is a reinforcement phenomenon present in a K-layered media system, regardless of the values of the one-way travel times. By this phenomenon, it is possible for rays to travel different paths and yet reach the seismic sensor at exactly the same time. In Ref. 1, ray diagrams are given for secondary reflections in a 3-layer media system. There are 14 distinct paths for the secondaries, but only 10 distinct times at which the secondaries will be recorded. The four times at which two secondaries add together (i.e., reinforce one another) are: $t = 4\tau_1 + 2\tau_2$, $4\tau_1 + 2\tau_2 + 2\tau_3$, $4\tau_1 + 4\tau_2 + 2\tau_3$, and $2\tau_1 + 4\tau_2 + 2\tau_3$. In our example, the reinforced secondaries occur at t = 0.836, 1.37, 1.406, and 1.006; those secondaries are starred in Figure 12. The situation is much the same for the tertiaries. The large negative tertiary at $t \cong 1.8$ is due to the reinforcement of a number of (as of yet unknown) tertiaries. We are presently studying

this reinforcement phenomenon to establish the times at which a reinforcement occurs as well as the number of components which add up to give the complete signal at those times.

VII. Conclusions

We have developed a canonical Bremmer Series decomposition of a complete seismogram for a K-layer media system. This decomposition is depicted in Figure 10. Our algorithms for computing the constituents, $y_1(t), y_2(t), \ldots$, etc., of the complete seismogram are state space primaries, secondaries, ..., etc. models, respectively. These algorithms are summarized in the theorem given in Section V. We have shown that each one of our constituent models only has zeros, whereas, as is well known, the complete model has both zeros and poles. This property of our constituent models makes them easy to use for computation purposes; and, in many situations the Bremmer Series can be truncated after a relatively small number of terms. Hence, the canonical Bremmer Series decomposition can also be viewed as a computational technique for approximately solving the lossless wave equation.

We conclude by relating our algorithms for computing the constituents $y_1(t), y_2(t), \ldots$ to those given by Bremmer (Ref. 2). In Bremmer's notation, u_j denotes the constituent of y associated with exactly j reflections; hence, his $u_1, u_3, u_5, \ldots$ should be associated with our $y_1, y_2, y_3, \ldots$. Bremmer's equations for his constituents are given as a function of spatial variable x, where $x = 0$ at the bottom of the layered media and x increases in the direction of the basement to the surface.

Bremmer's equations, from which we can calculate $u_1, u_3, u_5, \ldots$, are given by the following recursive integrals:

$$u_0(x) = \left[\frac{k_0}{k(x)}\right]^{\frac{1}{2}} \exp\left\{i\int_0^x k(s)\, ds\right\} \tag{36a}$$

$$u_{2i+1}(x) = -\frac{1}{2[k(x)]^{\frac{1}{2}}}\int_x^{\infty} ds\, \frac{k'(s)}{[k(s)]^{\frac{1}{2}}}\, u_{2i}(s) \exp\left\{i\int_x^s k(\sigma)\, d\sigma\right\} \tag{36b}$$

$$u_{2i}(x) = \frac{1}{2[k(x)]^{\frac{1}{2}}}\int_0^x ds\, \frac{k'(s)}{[k(s)]^{\frac{1}{2}}}\, u_{2i-1}(s) \exp\left\{i\int_s^x k(\sigma)\, d\sigma\right\} \tag{36c}$$

where $i = 0, 1, 2, \ldots$; k_0 denotes the constant wave number of the homogeneous half space $x < 0$; and $k(x)$ represents the variable wave number in an inhomogeneous space $x > 0$.* Observe that u_1 is calculated from knowledge of u_0; u_3 is calculated from knowledge of u_2, which is calculated from knowledge of u_1; hence, u_3 is calculated from knowledge of u_1; u_5 is calculated from knowledge of u_3, etc. Once the primaries, u_1, have been computed, we can combine Eqs. (36b) and (36c) so that we can view u_{2i+1} as being calculated from knowledge of u_{2i-1} $(i = 1, 2, \ldots)$. This dichotomy between the algorithm used for computing the primaries and the algorithm used for computing all other constituents also appears in our Section V algorithm.

In our development, we have only been interested in the surface seismogram, y(t). In Bremmer's development, the surface is at $x = 0$; hence, his constituents $u_1(x), u_3(x), \ldots$, are related to our constituents $y_1(t), y_2(t), \ldots$, by the relationship that $y_j(t) = u_{2i-1}$ $(x = 0)$, $j = 1, 2, \ldots$. Just as he has related u_{2i+1} to u_{2i-1}, we have related our upgoing state vectors (see Figure 10) $\underline{u}_n$ to $\underline{u}_{n-1}$, so that $y_n(t)$ can be computed using knowledge of $\underline{u}_{n-1}$. His recursive relationships involve the constituents directly, whereas our recursive relationships involve the states for successive partial residual models.

It would be interesting to explore the relationships between Bremmer's algorithm and ours in more detail. This remains to be done.

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* These equations were derived by replacing the inhomogeneous space $x > 0$ by a set of homogeneous layers $0 < x < x_1$, $x_1 < x < x_2$, $x_2 < x < x_3, \ldots$ with the successive constant wave numbers $k_1, k_2, k_3, \ldots$. At a later point, Bremmer passes from this discontinuous medium to a continuously changing one by making the thickness $\Delta x_s = x_s - x_{s-1}$ of the various layers infinitely small.

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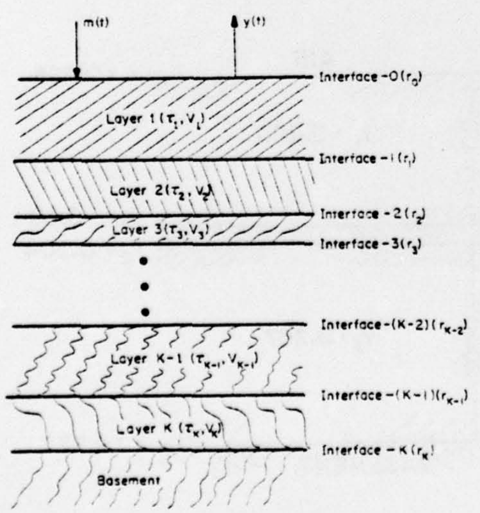

Figure 1 System of K-layered media.

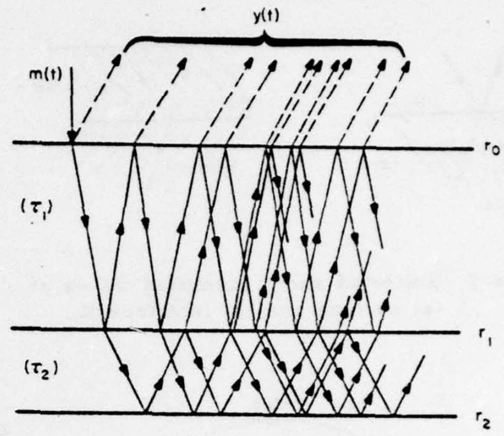

Figure 2 Two layer example.

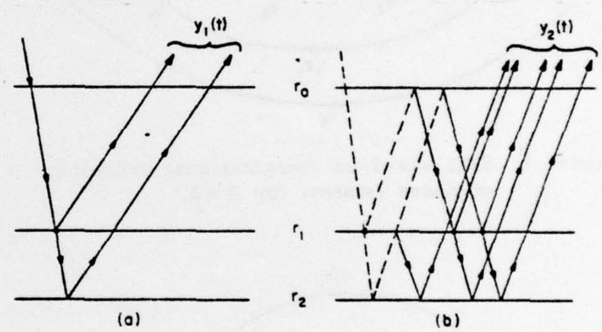

Figure 3 Primaries (a) and secondaries (b) for a two layer media. In (b), dashed lines denote the primaries which cause respective components of the secondaries.

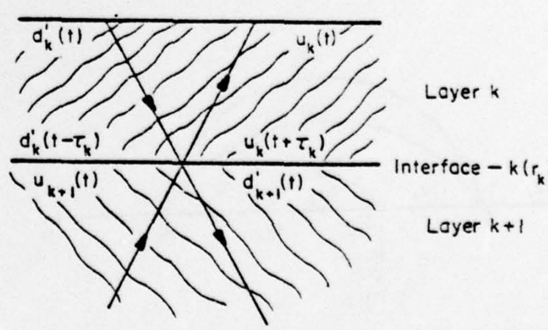

Figure 4 Reflected and transmitted waves at interface k. From Equation (11), $d'_k(t-\tau_k) \triangleq d_k(t)$.

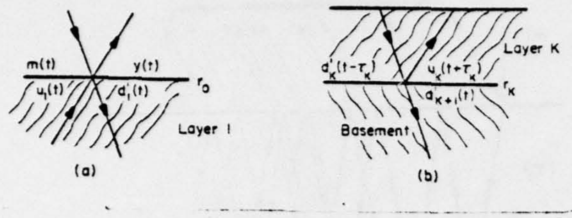

Figure 5 Reflected and transmitted waves at (a) surface and (b) interface K.

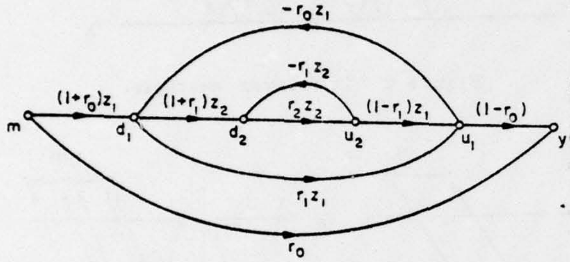

Figure 6 SFG based on computational ordering of nodes (states) for K = 2.

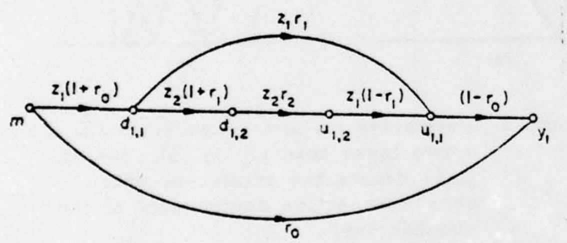

Figure 7 SFG for primaries model, for K = 2.

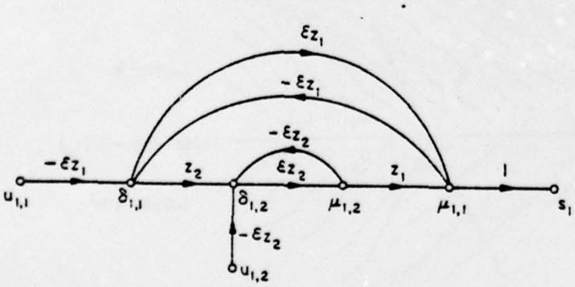

Figure 8 Proportionality SFG for first residual model, for K = 2. This SFG is obtained using Equations (24) and (25) for K = 2.

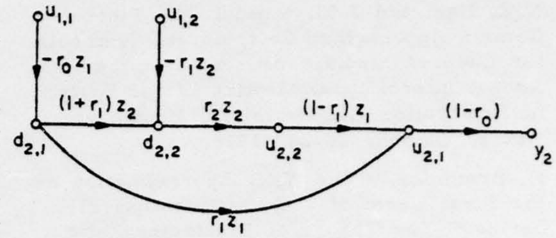

Figure 9 SFG for secondaries model, for K = 2.

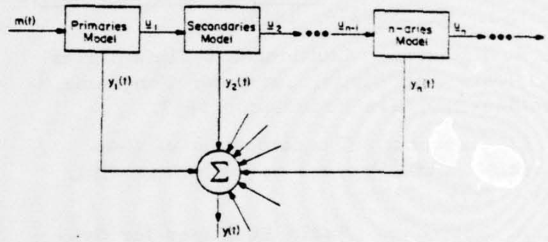

Figure 10 Canonical Bremmer Series decomposition of a seismogram signal, y(t). Vector $\underline{u}_n$ denotes the collection of K upgoing states for the n-aries model.

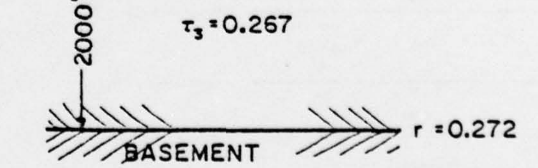

Figure 11 Three layer bright spot example.

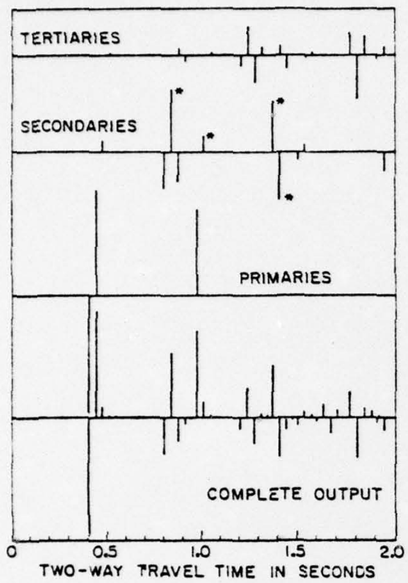

Figure 12 Responses for bright spot example.